Lecture 9 Risk Aversion, Investment and Saving

(2019.3.18)

<u>Recap</u>: If $\succsim$ on $\mathcal{L}$ (space of lottery) is rational (complete, transitive), continuous and satisfies independence axiom

$$\Rightarrow U(L) = \sum_{n=1}^{N} p_n u(x_n) \qquad \text{(EU)}$$

- Risk aversion as a result of decreasing marginal utility
  $\Rightarrow$ curvature of utility function as a measurement of risk aversion
- $R_A(y) = -\dfrac{u''(y)}{u'(y)}$ ARA

  $R_R(y) = -\dfrac{y\,u''(y)}{u'(y)}$ RRA
- CARA, CRRA, risk neutral
- Mean-Variance as a special case of EU

9.1 Conditions for buying risky assets

$$\begin{array}{cc} \tilde{r} & r_f \\ (a\ , & w_0 - a) \end{array} \qquad \text{when } a > 0\,?$$

One period optimization

$$\max_a E[u(\tilde{w})] = \max_a E\{u[w_0(1+r_f) + a(\tilde{r} - r_f)]\}$$

FOC: $E\{u'[w_0(1+r_f) + a^*(\tilde{r} - r_f)](\tilde{r} - r_f)\} = 0$

<u>Prop. 9.1</u> (i) $a^* > 0 \iff E(\tilde{r}) > r_f$

(ii) $a^* = 0 \iff E(\tilde{r}) = r_f$

(iii) $a^* < 0 \iff E(\tilde{r}) < r_f$

<u>Proof</u>: Define value function

$$V(a) = E\{u[w_0(1+r_f) + a(\tilde{r} - r_f)]\}$$

FOC can be rewritten as

$$V'(a^*) = E\{u'[W_0(1+r_f)+a^*(\tilde{r}-r_f)](\tilde{r}-r_f)\}=0$$

$$V''(a) = E\{u''[W_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)^2\}<0$$

$\Rightarrow V'(a)$ is a decreasing function

$\therefore a^*>0 \Leftrightarrow V'(0)>0$

(Because if $V'(0)<0$, $V'(a)<0$ for all $a>0$,)
$V'(a^*)=0$ cannot hold for $a^*>0$.

$$V'(0) = E\{u'[W_0(1+r_f)](\tilde{r}-r_f)\}$$
$$= u'[W_0(1+r_f)][E(\tilde{r})-r_f]$$

$V'(0)>0 \Leftrightarrow E(\tilde{r})>r_f$

That proofs (i). (ii) and (iii) can be proven similarly.

Remarks: $a=0 \rightarrow a>0$ brings 2 effects.

(1) EU↑ due to higher return (at the order of a)

(2) EU↓ due to higher risk (at the order of a^2)

when a is small, a dominates a^2.

(Appendix 9.A Arrow-Pratt Approximation)

9.2 Amount of Wealth in Risky Asset

Proposition 9.2: $E(\tilde{r})>r_f$, $u''(\cdot)<0$

(i) $a^{*\prime}(W_0)>0 \Leftrightarrow R_A'(\cdot)<0$ (DARA)

(ii) $=$ $\quad$ $=$ (CARA)

(iii) $<$ $\quad$ $>$ (IARA)

Proof of $\Leftarrow$ part of (i):

FOC $\quad E[u'(\tilde{w})(\tilde{r}-r_f)]=0$

$$\tilde{w} = W_0(1+r_f)+a^*(\tilde{r}-r_f)$$

Derivating w.r.t. W_0

$\Rightarrow E\left\{u''(\tilde{w})(\tilde{r}-r_f)\left[(1+r_f)+(\tilde{r}-r_f)\dfrac{da^*}{dw_0}\right]\right\}=0$

$\Rightarrow (1+r_f)E[u''(\tilde{w})(\tilde{r}-r_f)] + E[u''(\tilde{w})(\tilde{r}-r_f)^2]\dfrac{da^*}{dw_0}=0$

(Note that $\dfrac{da^*}{dw_0}$ is NOT a random variable)

$$\Rightarrow \frac{da^*}{dw_0} = -\frac{(1+r_f)E[u''(\tilde{w})(\tilde{r}-r_f)]}{E[u''(\tilde{w})(\tilde{r}-r_f)^2]}$$

As the denominator is always negative, the sign of $\dfrac{da^*}{dw_0}$ is determined by the sign of $E[u''(\tilde{w})(\tilde{r}-r_f)]$.

$E[u''(\tilde{w})(\tilde{r}-r_f)]$

$= E[-u'(\tilde{w})R_A(\tilde{w})(\tilde{r}-r_f)]$

$= \sum_{n=1}^{N} p_n[-u'(w_n)]R_A(w_n)(r_n-r_f)$

For a state n, either $r_n \geq r_f$ or $r_n \leq r_f$.

If $r_n \geq r_f$, as $a^*>0$ (follows $E(\tilde{r})>r_f$),

$w_n \geq w_0(1+r_f)$

DARA $\Rightarrow R_A(w_n) \leq R_A[w_0(1+r_f)]$

$\Rightarrow R_A(w_n)(r_n-r_f) \leq R_A[w_0(1+r_f)](r_n-r_f)$

It can be proven that this inequality also holds for $r_n \leq r_f$.

Therefore

$R_A(w_n)(r_n-r_f) \leq R_A[w_0(1+r_f)](r_n-r_f), \quad \forall n$

$\Rightarrow [-u'(w_n)]R_A(w_n)(r_n-r_f) \geq [-u'(w_n)]R_A[w_0(1+r_f)](r_n-r_f), \ \forall n$

for some n, $r_n \neq r_f$ (otherwise $\tilde{r}$ is r_f)

Therefore.

$$\sum_{n=1}^{N} p_n[-u'(w_n)] R_A(w_n)(r_n - r_f) > \sum_{n=1}^{N} p_n[-u'(w_n)] R_A[w_0(1+r_f)](r_n - r_f)$$

$$= E\{[-u'(\tilde{w})] R_A[w_0(1+r_f)](\tilde{r} - r_f)\}$$

$$= -R_A[w_0(1+r_f)] E[u'(\tilde{w})(\tilde{r} - r_f)]$$

$$= 0 \quad (\because \text{FOC})$$

$\therefore$ DARA $\Rightarrow E[u''(\tilde{w})(\tilde{r} - r_f)] > 0 \Rightarrow \dfrac{da^*}{dw_0}$.

Remarks: • Discussion of FOC is fruitful.

• $E[\cdot] \rightarrow \sum p_n(\cdot)$ useful technique.

9.3 Share of Risky Assets in Total Wealth

$$e(w_0) \triangleq \frac{da^*}{a^*} \Big/ \frac{dw_0}{w_0} = \frac{w_0}{a^*} \cdot \frac{da^*}{dw_0}$$

Proposition 9.3 $\quad E(\tilde{r}) > r_f$, $u''(\cdot) < 0$

(i) $e(w_0) > 1 \iff R_R'(\cdot) < 0$ (DRRA)

(ii) $\quad$ 〃 $= 1 \iff$ 〃 $= 0$ (CRRA)

(iii) $\quad$ 〃 $< 1 \iff$ 〃 > 0 (IRRA)

With CRRA utility, share of risky assets in total wealth is a constant, which is in line with empirical findings. Therefore, CRRA is more widely used in economic analysis.

Question: If preference is DRRA (or IRRA), what we shall see in the real world along the path of economic growth?

9.4 Risk-neutral investors

$u(c) = \alpha c$, $u'(c) = \alpha$, $u''(c) = 0$.

$$\max_a E[u(\tilde{w})] = \max_a E\{\alpha w_0(1+r_f) + \alpha a(\tilde{r} - r_f)\}$$

$$= \alpha w_0(1+r_f) + \alpha \max_a \{a[E(\tilde{r}) - r_f]\}$$

If $E(\tilde{r}) > r_f$, $a \to +\infty$.

9.5 Saving under Risk

9.5.1 Saving without uncertainty

$$\max_s u(w-s) + \delta u(sR)$$

FOC: $u'(w-s) = \delta R u'(sR)$

Derivating both sides w.r.t R .

$$-u''(w-s)\frac{ds}{dR} = \delta u'(sR) + \delta R u''(sR)(s + R\frac{ds}{dR})$$

$$\Rightarrow \frac{ds}{dR} = \frac{\delta u'(sR) + \delta s R u''(sR)}{-u''(w-s) - \delta R u''(sR)}$$

The denominator is always positive (as $u'' < 0$)

$$\text{numerator} = \delta u'(sR)\left[1 + \frac{sRu''(sR)}{u'(sR}\right]$$

$$= \delta u'(sR)[1 - R_R(sR)]$$

9.5.2 Remarks

$R\uparrow$ → Income effect $s\downarrow$
$R\uparrow$ → Substituting effect $s\uparrow$

$\frac{ds}{dR}$ is determined by which effect plays a dominating role. And that is in turn determined by whether $R_R > 1$ or $R_R < 1$.

$R_R < 1 \Rightarrow \frac{ds}{dR} > 0 \Rightarrow$ Substituting effect dominates.

$R_R > 1 \Rightarrow \frac{ds}{dR} < 0 \Rightarrow$ Income " "

Why is saving behavior (even under certainty) determined by risk aversion (RRA)?

- Intertemporary choice

$$\max_{w_1, w_2} u(w_1) + \delta u(w_2) \quad \text{s.t. } w_1 + w_2 = w$$

- Choice under uncertainty

$$\max_{w_1, w_2} p_1 u(w_1) + p_2 u(w_2) \quad \text{s.t. } w_1 + w_2 = w$$

- FOC:

$$u'(w_1) = \delta u'(w_2)$$

$$p_1 u'(w_1) = p_2 u'(w_2)$$

- Similar form, similar economic meaning.

To maximize utility, one should smooth consumption among different time/state. The strength of incentive to smooth consumption is governed by risk aversion (RRA)

- Two economic forces (time smoothing / state smoothing), one parameter to capture — something might go wrong in the analysis!

To be revisited in "Equity Premium Puzzle".

9.5.3 Saving under Uncertainty

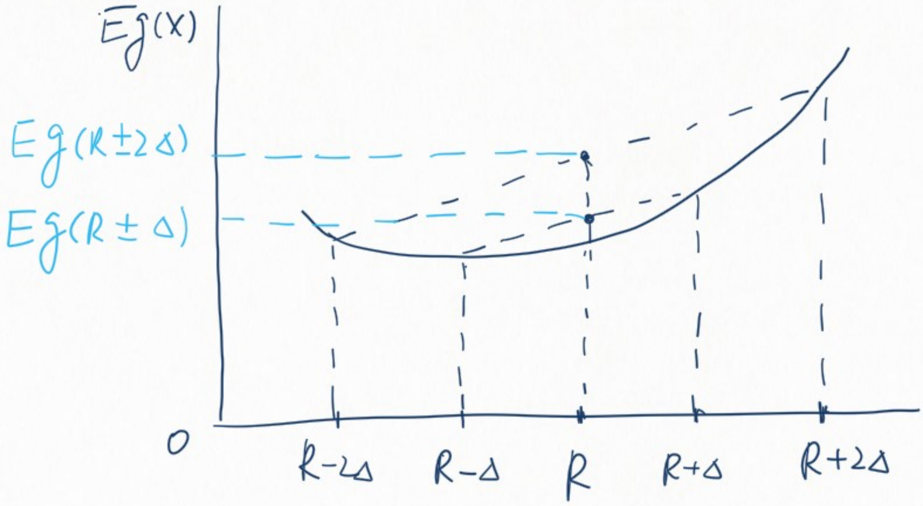
$$\max_{s} u(w-s) + \delta E[u(sR)]$$

FOC: $u'(w-s) = \delta E[R u'(sR)]$

LHS is a increasing function w.r.t. S.

$$g(R) \triangleq R u'(SR)$$

We need $g(R)$ to be a convex function ($g''>0$) in order to have that $\sigma^2(R)\uparrow \Rightarrow S\uparrow$



$$g'(R) = u'(SR) + SRu''(SR)$$

$$g''(R) = 2Su''(SR) + S^2 R u'''(SR)$$

$$P_R(y) \triangleq -\frac{y u'''(y)}{u''(y)}$$ Coefficient of relative Prudence (Kimball 1990)

$$g''(R) = Su''(SR)\left[2 + \frac{SRu'''(SR)}{u''(SR)}\right]$$

$$= Su''(SR)\left[2 - P_R(SR)\right]$$

$$P_R(SR) > 2 \Rightarrow g''(R) > 0.$$

Proposition 9.4 R_B is more risky than R_A.

(i) $S_A > S_B \iff P_R(SR) < 2$

(ii) $=$ $=$

(iii) $<$ $>$

More prudence, higher saving in more risky environment.

CRRA $\quad u(c) = \dfrac{c^{1-\gamma} - 1}{1-\gamma}$

$$u'(c) = c^{-\gamma}$$

$$u''(c) = -\gamma c^{-\gamma-1}$$

$$u'''(c) = \gamma(\gamma+1) c^{-\gamma-2}$$

$P_R(c) = \gamma + 1$. constant prudence.

log utility $\gamma = 1$.

$$R_R(c) = 1$$

$$P_R(c) = 2$$

saving is NOT affected by R (both its mean and its riskiness).

9.6 Final Remarks

Our intuition is backed, reinforced and enhanced by the theory — power of the theory!