Lecture 8 Expected Utility (2019. 3. 16)

8.1 From CAPM to General Equilibrium

	CAPM	C-CAPM
Preference	Mean-Variance	Expected utility (Lec 8)
Behavior	Portfolio optimization	Decision under uncertainty (9)
Equilibrium	Partial (asset market)	General (whole economy) (10)
Asset Pricing	CAPM (SML)	C-CAPM (11, 12)

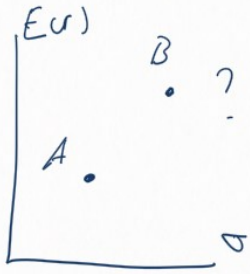
- **Problems regarding M-V preference**

, Rational Preference ($X \subset R_+^M$ choice set)

(i) complete $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$ holds (or both hold)

(ii) Transitivity $x \succsim y,\ y \succsim z \Rightarrow x \succsim z$

, Mean-Variance preference is not complete.



$u(r) = E(r) - A\sigma^2(r)$

Where does this utility function come from?

, For any random variable x.

$E(x-c)^k$ Moment of order k (k is a positive integer)

Mean : $k=1,\ c=0$

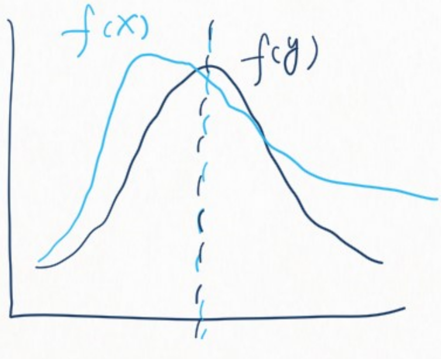
Variance : $k=2,\ c=E(x)$

In Mean-Variance preference, information of higher orders is lost.

$k=3$: skewness

$k=4$: kurtosis

$\vdots$



- **Problems regarding partial equilibrium**

  Individual asset returns are linked to market return in CAPM. But we can't say anything about underlying driving forces of the market — our analysis stops at the assumptions.

- We need a more general framework to get a deeper understanding.

8.2 Expected Utility (EU)

8.2.1 St. Petersberg Pardox

$$\text{Expected Payoff} = \frac{1}{2}\times 1 + \left(\frac{1}{2}\right)^2 \times 2 + \left(\frac{1}{2}\right)^3 \times 4 + \cdots$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot 2^{n-1} = \frac{1}{2}\sum_{n=1}^{\infty} 1^n = +\infty.$$

$$\text{Expected Utility} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot \ln(2^{n-1})$$

$$= \ln 2 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot (n-1)$$

$$\begin{aligned} = \ln 2 \cdot \Big(& \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \\ & + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\ & + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \\ & + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots \Big) \end{aligned}$$

$$= \ln 2 \cdot \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right)$$

$$= \ln 2 < +\infty$$

With logrithm utility, people will only pay $2 \,(= e^{\ln 2})$ at most to join the gamble.

- Continuous preference

$$x^n \succsim y^n \ \forall n, \quad \lim_{n\to\infty} x^n \succsim \lim_{n\to\infty} y^n$$

- Proposition 8.1: A rational and continuous preference can be represented by a continuous function.

$$x \succsim y \iff u(x) \geq u(y)$$

8.2.3 EU Theory

Step 1: modeling choice set under uncertainty

Step 2: Describing preference under uncertainty

Step 3: Finding out a utility function

- Without uncertainty, elements in the choice set are consumption bundles.

Apple	Pear	
(2 ,	1)	$\in X \subset R_+^2$
(1 ,	2)	$\in X \subset R_+^2$

- Under uncertainty, elements are probabilities of different consumption bundles.

- Definition 8.3 (Simple lottery)

$$L = (p_1, \cdots, p_N). \quad p_n \geq 0 \ (\forall n), \ \sum_{n=1}^{N} p_n = 1.$$

Possible outcomes are certain consumption bundles.

- Definition 8.4 (Compound lottery)

Possible outcomes are lotteries.

compound lotteries can be represented as simple lotteries.

Example $A = (2 \text{ Apples}, 1 \text{ pear})$
$B = (1 \text{ Apple}, 2 \text{ Pears})$

$L_1 = (0.5, 0.5)$, $L_2 = (0.25, 0.75)$

Compound lottery

$(L_1, L_2; 0.5, 0.5) = (0.375, 0.625)$

$= 0.5 \times 0.5 + 0.5 \times 0.25$

$= 0.5 \times 0.5 + 0.5 \times 0.75$

- $\mathcal{L}$ = space of all (simple) lotteries
- Definition 8.5 (Independence Axiom)

  $A, B, C \in \mathcal{L}$, $\forall \alpha \in [0, 1]$

  If $A \succsim B \Leftrightarrow \alpha A + (1-\alpha) C \succsim \alpha B + (1-\alpha) C$

  Then we say $\succsim$ satisfies Independence Axiom (IA)
- Indepence Axiom is not that innocent as it seems.

  A = beef. B = bitter melon (苦瓜). C = pork.

  $A \succ B$, but not necessarily

  $\alpha A + (1-\alpha) C \succ \alpha B + (1-\alpha) C$.

  But we need IA to have expected utility function.
- Proposition 8.2 (Expected Utility Theorem)

  $\succsim$ on $\mathcal{L}$ rational, continuous, IA

  $$U(L) = \sum_{n=1}^{N} p_n u(x_n)$$
- We can use EU to order any two different lotteries.

  For some lotteries, order can be given regardless of utility (all the people will make the same choice).

  — stochastic dominance (Appendix 8.A)

- Allais Paradox

	u_{25}	u_5	u_0
	win 2.5 million	win 0.5 million	broken legs
$L_1 =$	(0 ,	1 ,	0)
$L_1' =$	(0.1 ,	0.89 ,	0.01)
$L_2 =$	(0 ,	0.11 ,	0.89)
$L_2' =$	(0.1 ,	0 ,	0.9)

For most people $L_1 \succ L_1'$, $L_2' \succ L_2$

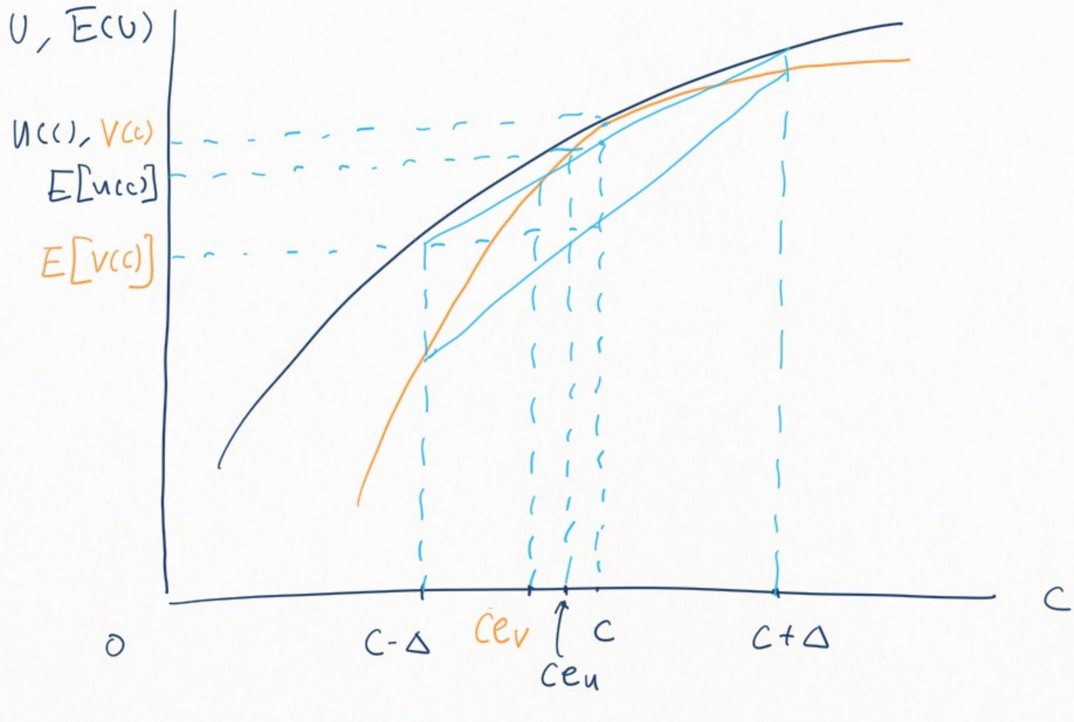
$$u_5 > 0.1 u_{25} + 0.89 u_5 + 0.01 u_0 \quad (L_1 \succ L_1')$$

$(+0.89\, u_0 - 0.89\, u_5$ at both sides)

$$0.11 u_5 + 0.89 u_0 > 0.1 u_{25} + 0.9 u_0 \quad (L_2 \succ L_1' \;!)$$

- Despite the Allais Paradox and other anormalies, EU is still the main workhorse in economic analysis under uncertainty (NO better alternatives!)

8.3 Measurment of Risk Aversion



risk premium = c − ce (certainty equivalence)

affected by c and σ (as well as curvature of u, v)

we need a measurment of risk aversion only affected by utility function.

8.3.2 Coefficient of Absolute Risk Aversion (ARA)

$$\underbrace{u(y)}_{\text{certainty}} = \underbrace{\pi^* u(y+h) + (1-\pi^*)\, u(y-h)}_{\text{gamble}}$$

π^* — probability of win in the gamble that makes the person indifferent.

More risk averse → higher π^*

Taylor expansion to order 2 (It is important to get to order 2)

$$u(y+h) = u(y) + h u'(y) + \frac{h^2}{2} u''(y) + \cdots$$

$$u(y-h) = u(y) - h u'(y) + \frac{h^2}{2} u''(y) + \cdots$$

$$\cancel{u(y)} = \pi^* \left[\cancel{u(y)} + h u'(y) + \frac{h^2}{2} u''(y)\right]$$

$$+ (1-\pi^*)\left[\cancel{u(y)} - h u'(y) + \frac{h^2}{2} u''(y)\right]$$

$$\Rightarrow \quad 0 = (2\pi^* - 1) h u'(y) + \frac{h^2}{2} u''(y)$$

$$\Rightarrow \quad \pi^* = \frac{1}{2} + \frac{h}{4}\left[-\frac{u''(y)}{u'(y)}\right]$$

$$R_A(y) \triangleq -\frac{u''(y)}{u'(y)} \qquad \text{(ARA)}$$

People usually have different ARA at different consumption levels.

8.3.3 Coefficient of Relative Risk Aversion (RRA)

$$u(y) = \pi^* u(y+\theta y) + (1-\pi^*) u(y-\theta y)$$

$$h = \theta y$$

$$\Rightarrow \pi^* = \frac{1}{2} + \frac{\theta y}{4}\left[-\frac{u''(y)}{u'(y)}\right] = \frac{1}{2} + \frac{\theta}{4}\left[-\frac{y u''(y)}{u'(y)}\right]$$

$$R_R(y) \triangleq -\frac{y u''(y)}{u'(y)} \quad \text{(RRA)}$$

8.3.4 Commonly used utility functions

CARA: $u(c) = -e^{-\alpha c}$. $R_A(c) = \alpha$

CRRA: $u(c) = \dfrac{c^{1-\gamma} - 1}{1-\gamma}$ $R_R(c) = \gamma$

$u(c) = \ln c$ (when $\gamma = 1$)

Linear (Risk neutral)

$u(c) = \alpha c$, $R_A(c) = R_R(c) = 0$

8.4 M-V preference as a special case of EU

- Quadratic utility function
- CARA + lognormal return

$z = \ln x \sim N(\mu, \sigma^2)$

$E(e^z) = E(x) = e^{\mu + \frac{1}{2}\sigma^2}$ (Appendix 8.B)

$u(c) = -e^{-\alpha c}$

$$E[u(c)] = E\left[-e^{-\alpha c}\right] = -e^{-\alpha E(c) + \frac{1}{2}\alpha^2 Var(c)}$$

$$\max E[u(c)] \Leftrightarrow \max E(c) - \frac{1}{2}\alpha^2 Var(c)$$