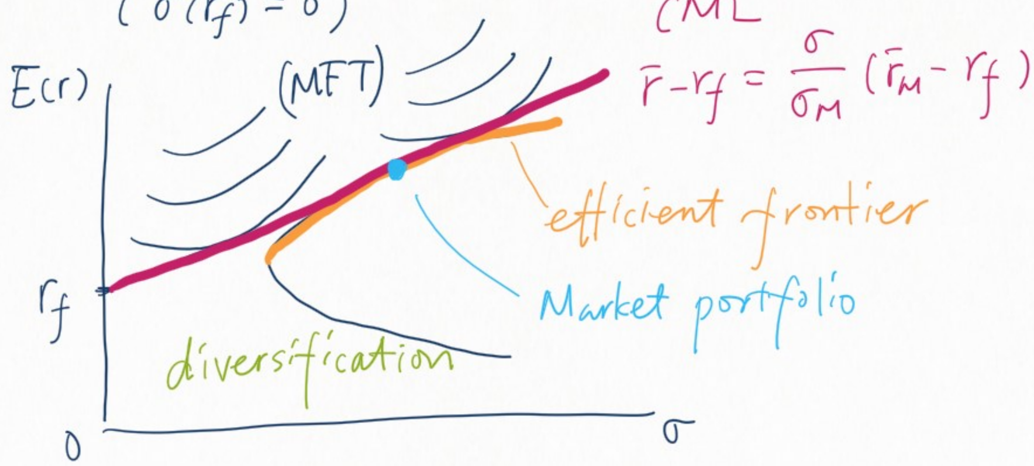
Lecture 6 CAPM (2019.3.4)

- Recap: Historical data
  $\Rightarrow$ Mean and Variance of ex-post rate of return
  $\rightarrow$ a sense of ex-ante rate of return $E(\tilde{r})$
  $(\sigma(r_f) = 0)$



6.1, 6.2 From Portfolio Selection to Market Equilibrium

- What does market portfolio (M) looks like?
  — M is the entire market for all risky assets!
- How can it be that the M as a result of a complicated portfolio optimization problem to be precisely identical to the entire market?
  — Everyone holds M. If M is not identical to the entire market, there must be some assets that supply $\neq$ demand. Then, asset prices will adjust to clear the market.
- M = entire market as a result of price mechanism.
  — When the market is cleared, asset prices should satisfy some propeties (CAPM)

- Expected mean and variance of r (implied by asset prices)
  - $\Rightarrow$ Portfolio selection (M)
  - $\Rightarrow$ Asset prices (CAPM)

circular reasoning ?

— Mutual causality in Equilibrium

Asset prices $\rightleftarrows$ portfolio selection

(determined simultaneously)

- **Assumptions of CAPM**

(1) No transaction costs
(2) No taxes
(3) Infinitely divisible
(4) Perfect competition

} simplify assumptions about the market

(5) Mean-Variance preference
(6) No limit on short-selling
(7) Common expectation

} Mean-Variance optimization

- **Two ways to derive CAPM**

"Everyone holds M in equilibrium" means

(i) Holding any portfolio other than M can NOT yield higher utility for anyone.

(ii) Other portfolio can NOT offer better mean-variance composition than M.

6.3.2 1st Proof of CAPM (based on utility)

$u(r) = E(r) - A\sigma^2(r) \qquad A > 0$

He only holds M (an assumption can be relaxed)

A new portfolio $p = (\underset{i}{w}, \underset{M}{1-w})$

$$u(r_p) = u[wr_i + (1-w)r_M]$$
$$= E[wr_i + (1-w)r_M] - A\sigma^2[wr_i + (1-w)r_M]$$
$$= wE(r_i) + (1-w)E(r_M)]$$
$$- A[w^2\sigma_i^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{iM}]$$
$$= wE(r_i) + (1-w)E(r_M) - Aw^2(\sigma_i^2 + \sigma_M^2 - 2\sigma_{iM})$$
$$- 2Aw(\sigma_{iM} - \sigma_M^2) - A\sigma_M^2$$

$$\frac{du(r_p)}{dw} = E(r_i) - E(r_M) - 2Aw(\sigma_i^2 + \sigma_M^2 - 2\sigma_{iM}) - 2A(\sigma_{iM} - \sigma_M^2)$$

$$\left.\frac{du(r_p)}{dw}\right|_{w=0} = 0$$ (otherwise holding more i will yield higher utility for the investor)

$$\Rightarrow E(r_i) - E(r_M) - 2A(\sigma_{iM} - \sigma_M^2) = 0 \quad \ldots\ldots (*)$$

Equation (*) holds for any asset (including r_f)

$$\Rightarrow r_f - E(r_M) + 2A\sigma_M^2 = 0 \Rightarrow A = \frac{E(r_M) - r_f}{2\sigma_M^2}$$

Substituting it into (*) yields

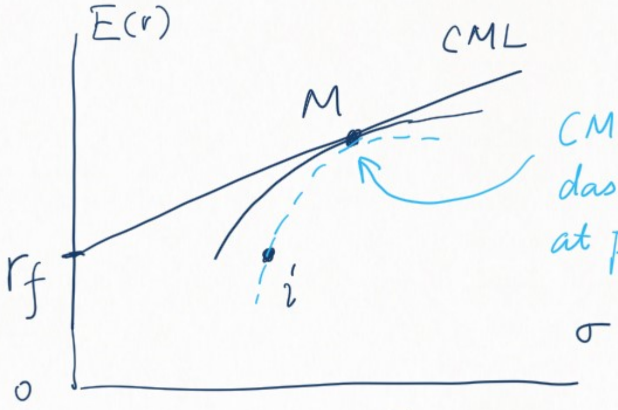
$$E(r_i) - E(r_M) - \frac{E(r_M) - r_f}{\sigma_M^2}(\sigma_{iM} - \sigma_M^2) = 0$$

$$\Rightarrow E(r_i) - r_f = \frac{\sigma_{iM}}{\sigma_M^2}[E(r_M) - r_f]$$

Define $\beta_i \triangleq \dfrac{\sigma_{iM}}{\sigma_M^2}$

$$\boxed{E(r_i) - r_f = \beta_i[E(r_M) - r_f]}$$ <u>CAPM Equation</u>

6.4.1 2nd Proof of CAPM (based on Sharpe Ratio)



CML is tangent to the dashed hyperbolic curve at point M. Otherwise M CANNOT be the best portfolio everyone holds.

CML: $E(r) = r_f + \frac{E(r_M) - r_f}{\sigma_M}\sigma$

Slope of CML: $\frac{E(r_M) - r_f}{\sigma_M}$

Portfolio $\underset{i\quad\;\; M}{(w, 1-w)} \triangleq r_w$

$$E(r_w) = wE(r_i) + (1-w)E(r_M) = w[E(r_i) - E(r_M)] + E(r_M)$$

$$\sigma^2(r_w) = w^2\sigma_i^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{iM}$$

$$= w^2(\sigma_i^2 + \sigma_M^2 - 2\sigma_{iM}) + 2w(\sigma_{iM} - \sigma_M^2) + \sigma_M^2$$

$$\frac{dE(r_w)}{dw} = E(r_i) - E(r_M)$$

$$\frac{d\sigma(r_w)}{dw} = \frac{1}{2}\left[w^2(\sigma_i^2 + \sigma_M^2 - 2\sigma_{iM}) + 2w(\sigma_{iM} - \sigma_M^2) + \sigma_M^2\right]^{-\frac{1}{2}}$$

$$\times \left[2w(\sigma_i^2 + \sigma_M^2 - 2\sigma_{iM}) + 2(\sigma_{iM} - \sigma_M^2)\right]$$

Slope of the dashed curve at point M

$$\left.\frac{dE(r_w)}{d\sigma(r_w)}\right|_{w=0} = \left[\frac{dE(r_w)}{dw} \Big/ \frac{d\sigma(r_w)}{dw}\right]_{w=0}$$

$$= [E(r_i) - E(r_M)] \Big/ \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M} = \text{slope of CML}$$

$$\Rightarrow \frac{\sigma_M [E(r_i) - E(r_M)]}{\sigma_{iM} - \sigma_M^2} = \frac{E(r_M) - r_f}{\sigma_M}$$

$$\Rightarrow E(r_i) - r_f = \frac{\sigma_{iM}}{\sigma_M^2}[E(r_M) - r_f]$$

Define $\beta_i \triangleq \dfrac{\sigma_{iM}}{\sigma_M^2}$

$$\boxed{E(r_i) - r_f = \beta_i [E(r_M) - r_f]}$$ CAPM Equation

6.4.2 Sharpe Ratio

$$SR_i = \frac{E(r_i) - r_f}{\sigma(r_i)}$$

measures the efficiency of acquiring higher rate of return by taking more risk.

$$SR_i = \frac{\bar{r}_i - r_f}{\sigma_i}$$ (in real world estimation)

Market portfolio M has the highest SR among all portfolios constructed with risky assets.
(A way to calculate M)

6.5 CML vs. SML

$E(r_i) = r_f + \beta_i [E(r_M) - r_f]$ Capital Market Line

$E(r_i) = r_f + \dfrac{\sigma_i}{\sigma_M}[E(r_M) - r_f]$ Securities Market Line

Expected rate of return = $\underbrace{\text{Time value of money}}_{r_f}$ + Risk premium

= measurement of risk x price of risk

How can CML and SML both be right?

- SML holds for all assets.
- CML only holds for efficient portfolios
  (with the highest SR)