Lecture 5 Mean-Variance Analysis 2019.3.2

Recap: • DDM $S_0 = \sum_{t=1}^{\infty} \frac{D_t}{(1+r)^t} + \lim_{t\to\infty} \frac{S_t}{(1+r)^t}$; TVC

Gordon $S_0 = \frac{D_1}{r-g}$

• Asset price = $\sum$ (discounted future payoffs)

discount rate ?

• Fisher separation theorem

(1) Max $S_1 = D_1 + \frac{D_2}{1+r}$

(2) Max U_i for all shareholders

Corporate behavior

← stock price (reflects the evaluation made by the market)

← behavior of shareholders and potential shareholders

← Some fundamental economic forces

(will be explained in detail in C-CAPM)

5.1 Introduction

Discount rate r ? ← behavior of investors

Markowitz 1952 Portfolio Selection

• What investors will do if they care about revenue (mean) and risk (variance)?

This question has fundamentally changed financial theories and the financial industry

— the 1st revolution in finance

Analogy of ordering in a restaurant

- Order dishes (according to tastes of each individual dishes)
- Order a feast (interactions of flavor of different dishes are important)

5.2 Some explanations on Mean and Variance

	Current price (period 0)		future payoff (period 1)	
riskfree asset	90	0.5	100	(state a)
		0.5	100	(state b)
risky asset	80	0.5	120	(a)
		0.5	80	(b)

- ex-ante rate of return (事前回报率)

  = expected rate of return (预期回报率)

  calculated at period 0 with uncertainty (Don't know which state will be realized in period 1)

riskfree $E(r_f) = \frac{0.5\times100+0.5\times100}{90} - 1 \doteq 11\%$

risky $E(\tilde{r}) = \frac{0.5\times120+0.5\times80}{80} - 1 = 25\%$

- ex-post rate of return (事后回报率)

  Calculated at period 1 without uncertainty

riskfree $r_{fa} = r_{fb} = \frac{100}{90} - 1 \doteq 11\%$

risky $r_a = \frac{120}{80} - 1 = 50\%$

$r_b = \frac{80}{80} - 1 = 0\%$

- $E(r_f) = r_{fa} = r_{fb}$ (riskfree)

  $E(\tilde{r}) = 0.5 r_a + 0.5 r_b$

- risk premium (风险溢价) only in expected r (NOT in ex-post r)

  risk premium $= E(\tilde{r}) - E(r_f)$

- In asset pricing, what really matters is expected r

$$X \rightarrow E(\tilde{r})$$

Use the mean of past ex-post r to estimate $E(\tilde{r})$.

Use the variance of past ex-post r to estimate the risk associated with $E(\tilde{r})$

- Mean and variance are calculated with historical data, but what investors really care is $E(\tilde{r})$

Question: What is the variance of riskfree rate r_f?

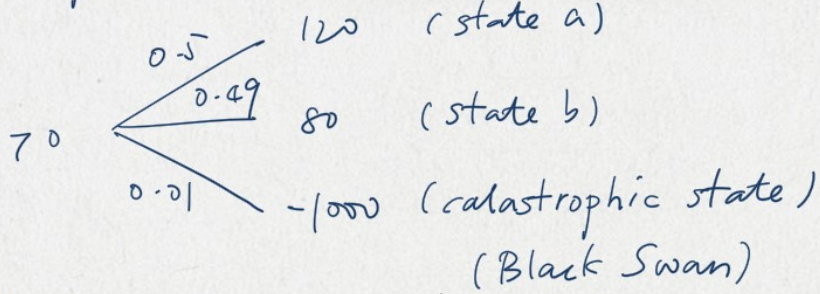
- NOT strictly speaking

$$\bar{r} = E(r) = \frac{1}{N}\sum_{i=1}^{N} r_i$$

$$\sigma_r^2 = E(r-\bar{r})^2 = \frac{1}{N}\sum_{i=1}^{N}(r_i-\bar{r})^2$$

$$\sigma_{xy} = E(x-\bar{x})(y-\bar{y}) = \frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})$$

- Survivorship bias



Misleading mean

$$\bar{r}_{misleading} = 0.5\times\left(\frac{120}{70}-1\right) + \underline{0.5}\times\left(\frac{80}{70}-1\right)$$

$$= 0.5\times 71\% + 0.5\times 14\% = 43\%$$

Actual mean

$$\bar{r}_{actual} = 0.5\times\left(\frac{120}{70}-1\right) + 0.49\times\left(\frac{80}{70}-1\right) + 0.01\times\left(\frac{-1000}{70}-1\right)$$

$$= 27\%$$

5.3 Mean-Variance of a Portfolio

Portfolio with n assets

$(w_1, w_2, \cdots, w_n)$ $\quad w_i$ — share of wealth in asset i

$$\sum_{i=1}^{n} w_i = 1$$

$w_i > 0$ long (做多) asset i

$w_i < 0$ short (做空) asset i

5.3.1 1 riskfree r_f + 1 risky $\tilde{r}_s$

$(1-w, \quad w)$

$$\bar{r}_p = E\left[(1-w) r_f + w \tilde{r}_s\right] = (1-w) r_f + w E(\tilde{r}_s) = (1-w) r_f + w \bar{r}_s$$

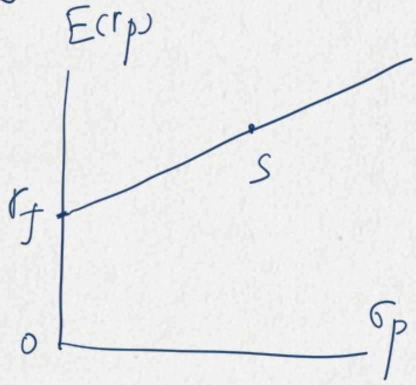
$$= r_f + w(\bar{r}_s - r_f)$$

$$\sigma_p^2 = E\left[\cancel{(1-w)r_f} + w\tilde{r}_s - \cancel{(1-w)r_f} - w\bar{r}_s\right]^2$$

$$= w^2 E\left[\tilde{r}_s - \bar{r}_s\right]^2$$

$$= w^2 \sigma_s^2 \quad \Rightarrow w = \frac{\sigma_p}{\sigma_s}$$

$$\therefore \bar{r}_p = r_f + \frac{\bar{r}_s - r_f}{\sigma_s} \sigma_p$$



5.3.2 1 risky $(\tilde{r}_1)$ + 1 risky $(\tilde{r}_2)$

$(w, \quad 1-w)$

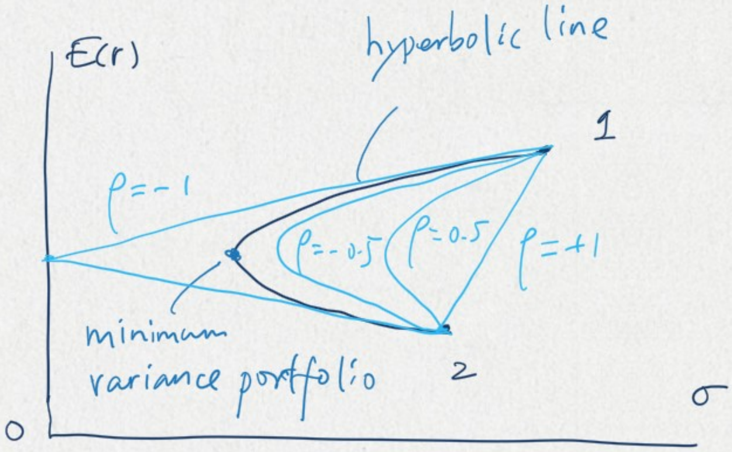
$$\bar{r}_p = E(\tilde{r}) = w\bar{r}_1 + (1-w)\bar{r}_2$$

$$\sigma_p^2 = E\left[w\tilde{r}_1 + (1-w)\tilde{r}_2 - w\bar{r}_1 - (1-w)\bar{r}_2\right]^2$$

$$= E\left[w(\tilde{r}_1 - \bar{r}_1) + (1-w)(\tilde{r}_2 - \bar{r}_2)\right]^2$$

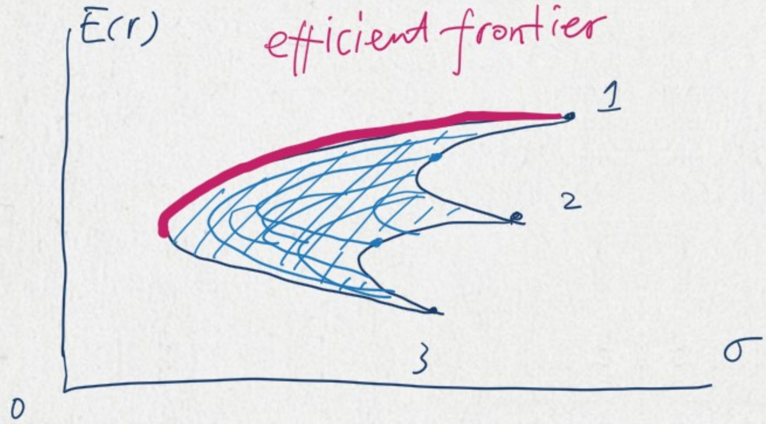
$$= E\left[w^2(\tilde{r}_1 - \bar{r}_1)^2 + (1-w)^2(\tilde{r}_2 - \bar{r}_2)^2 + 2w(1-w)(\tilde{r}_1 - \bar{r}_1)(\tilde{r}_2 - \bar{r}_2)\right]$$

$$= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}$$

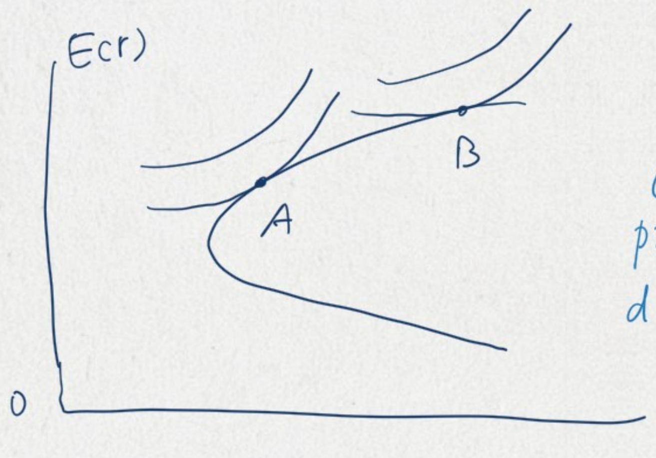


Benefit of diversification.

5.3.3 Efficient frontier of multi-assets

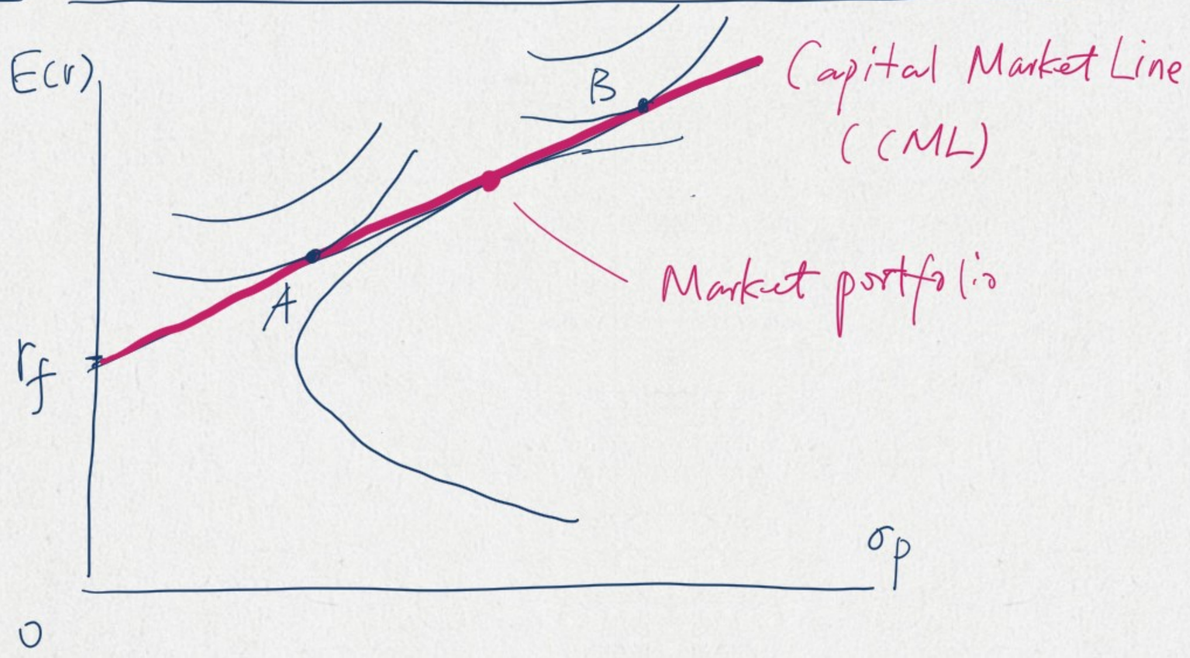


Investors' choices



It seems that different investors (with different) preferences will have different portfolio choices.

5.4 Market Portfolio and Mutual Fund Theorem



Market portfolio $(\bar{r}_M, \sigma_M)$

CML: $\bar{r} - r_f = \frac{\sigma}{\sigma_M}(\bar{r}_M - r_f)$

- All investors should hold the same portfolio of risky assets (market portfolio) regardless of their preferences

  — Mutual Fund Separation Theorem (MFT)

Remarks:

(1) MFT holds even there is no riskfree asset.

$\Rightarrow$ A portfolio of 2 risky assets on the efficient frontier lies on the efficient frontier

(2) MFT is the theoritical foundation of the mutual fund industry.